

A DENSITY HALES-JEWETT THEOREM FOR MATROIDS

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ABSTRACT. We show that, if $\alpha > 0$ is a real number, $n \geq 2$ and $\ell \geq 2$ are integers, and q is a prime power, then every simple matroid M of sufficiently large rank, with no $U_{2,\ell}$ -minor, no rank- n projective geometry minor over a larger field than $\text{GF}(q)$, and satisfying $|M| \geq \alpha q^{r(M)}$, has a rank- n affine geometry restriction over $\text{GF}(q)$. This result can be viewed as an analogue of the multidimensional density Hales-Jewett theorem for matroids.

1. INTRODUCTION

Furstenberg and Katznelson [3] proved the following result, implying that $\text{GF}(q)$ -representable matroids of nonvanishing density and huge rank contain large affine geometries as restrictions:

Theorem 1.1. *Let q be a prime power, $\alpha > 0$ be a real number, and n be a positive integer. If M is a simple $\text{GF}(q)$ -representable matroid with $|M| \geq \alpha q^{r(M)}$, and M has sufficiently large rank, then M has an $\text{AG}(n, q)$ -restriction.*

Later, Furstenberg and Katznelson [4] proved a much more general result; namely the Multidimensional Density Hales-Jewett Theorem, which gives a similar statement in the more abstract setting of words over an arbitrary finite alphabet. Considerably shorter proofs [1,13] have since been found. We will generalise Theorem 1.1 in a different direction:

Theorem 1.2. *Let q be a prime power, n be a positive integer, and $\alpha > 0$ be a real number. If M is a simple matroid with no $U_{2,q+2}$ -minor and $|M| \geq \alpha q^{r(M)}$, and M has sufficiently large rank, then M has an $\text{AG}(n, q)$ -restriction.*

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In fact, we prove more. The class of matroids with no $U_{2,q+2}$ -minor is just one of many minor-closed classes whose extremal behaviour is qualitatively similar to that of the $\text{GF}(q)$ -representable matroids. The following theorem, which summarises several papers [5,6,9], tells us that such classes occur naturally as one of three types:

Theorem 1.3 (Growth Rate Theorem). *Let \mathcal{M} be a minor-closed class of matroids, not containing all simple rank-2 matroids. There exists a real number $c_{\mathcal{M}} > 0$ such that either:*

- (1) $|M| \leq c_{\mathcal{M}} r(M)$ for every simple $M \in \mathcal{M}$,
- (2) $|M| \leq c_{\mathcal{M}} r(M)^2$ for every simple $M \in \mathcal{M}$, and \mathcal{M} contains all graphic matroids, or
- (3) there is a prime power q such that $|M| \leq c_{\mathcal{M}} q^{r(M)}$ for every simple $M \in \mathcal{M}$, and \mathcal{M} contains all $\text{GF}(q)$ -representable matroids.

We call a class \mathcal{M} satisfying (3) *base- q exponentially dense*. It is clear that these classes are the only ones that contain arbitrarily large affine geometries, and that the matroids with no $U_{2,q+2}$ -minor form such a class. Our main result, which clearly implies Theorem 1.2, is the following:

Theorem 1.4. *Let \mathcal{M} be a base- q exponentially dense minor-closed class of matroids, $\alpha > 0$ be a real number, and n be a positive integer. If $M \in \mathcal{M}$ is simple, satisfies $|M| \geq \alpha q^{r(M)}$, and has sufficiently large rank, then M has an $\text{AG}(n, q)$ -restriction.*

Finding such a highly structured restriction seems very surprising, given the apparent wildness of general exponentially dense classes. This will be proved using Theorem 1.3 and a slightly more technical statement, Theorem 6.1; the proof extensively uses machinery developed in [7], [8], [14] and [15].

We would like to prove a result corresponding to Theorem 1.4 for *quadratically dense* classes satisfying condition (2) of Theorem 1.3. The following is a corollary of the Erdős-Stone Theorem [2]:

Theorem 1.5. *Let $\alpha > 0$ be a real number and n be a positive integer. If G is a simple graph such that $|E(G)| \geq \alpha |V(G)|^2$ and $|V(G)|$ is sufficiently large, then G has a $K_{n,n}$ -subgraph.*

In light of this, we expect that the unavoidable restrictions of dense matroids in a quadratically dense class are the cycle matroids of large complete bipartite graphs.

Conjecture 1.6. *Let \mathcal{M} be a quadratically dense minor-closed class of matroids, $\alpha > 0$ be a real number, and n be a positive integer. If*

$M \in \mathcal{M}$ is simple, satisfies $|M| \geq \alpha r(M)^2$, and has sufficiently large rank, then M has an $M(K_{n,n})$ -restriction.

2. PRELIMINARIES

We follow the notation of Oxley [16]. For a matroid M , we also write $|M|$ for $|E(M)|$, and $\varepsilon(M)$ for $|\text{si}(M)|$, or the number of points in M . If $\ell \geq 2$ is an integer, we write $\mathcal{U}(\ell)$ for the class of matroids with no $U_{2,\ell+2}$ -minor.

The next theorem, a constituent of Theorem 1.3, follows easily from the two main results of [5].

Theorem 2.1. *There is a real-valued function $\alpha_{2,1}(n, \gamma, \ell)$ so that, if $\ell \geq 2$ and $n \geq 2$ are integers, $\gamma > 1$ is a real number, and $M \in \mathcal{U}(\ell)$ satisfies $\varepsilon(M) \geq \alpha_{2,1}(n, \gamma, \ell)\gamma^{r(M)}$, then M has a $\text{PG}(n-1, q)$ -minor for some $q > \gamma$.*

The next theorem is due to Kung [11].

Theorem 2.2. *If $\ell \geq 2$ and $M \in \mathcal{U}(\ell)$, then $\varepsilon(M) \leq \frac{\ell^{r(M)} - 1}{\ell - 1}$.*

We will sometimes use the cruder estimate $\varepsilon(M) \leq (\ell + 1)^{r(M)-1}$ for ease of notation, such as in the following easy corollary:

Corollary 2.3. *If $\ell \geq 2$ is an integer, $M \in \mathcal{U}(\ell)$, and $C \subseteq E(M)$, then $\varepsilon(M/C) \geq (\ell + 1)^{-r_M(C)} \varepsilon(M)$.*

Proof. Let \mathcal{F} be the collection of rank- $(r_M(C) + 1)$ flats of M containing C . We have $\varepsilon(M|F) \leq \frac{\ell^{r_M(C)+1} - 1}{\ell - 1} \leq (\ell + 1)^{r_M(C)}$ for each $F \in \mathcal{F}$. Moreover, $|\mathcal{F}| = \varepsilon(M/C)$, and $\varepsilon(M) \leq \sum_{F \in \mathcal{F}} \varepsilon(M|F)$; the result follows. \square

We apply both the above results freely.

The next result follows from [8, Lemma 3.1].

Lemma 2.4. *Let q be a prime power, $k \geq 0$ be an integer, and M be a matroid with a $\text{PG}(r(M) - 1, q)$ -restriction R . If F is a rank- k flat of M that is disjoint from $E(R)$, then $\varepsilon(M/F) \geq \frac{q^{r(M/F)+k} - 1}{q - 1} - q \frac{q^{2k} - 1}{q^2 - 1}$.*

3. CONNECTIVITY

A matroid M is *weakly round* if there is no pair of sets A, B with union $E(M)$, such that $r_M(A) \leq r(M) - 1$ and $r_M(B) \leq r(M) - 2$. This is a variation on *roundness*, a notion equivalent to infinite vertical connectivity introduced by Kung in [12] under the name of *non-splitting*. Our tool for reducing Theorem 1.4 to the weakly round case is the following, proved in [14, Lemma 7.2].

Lemma 3.1. *There is an integer-valued function $f_{3.1}(r, d, \ell)$ so that, for any integers $\ell \geq 2$ and $0 \leq d \leq r$, and real-valued function $g(n)$ satisfying $g(d) \geq 1$ and $g(n) \geq 2g(n-1)$ for all $n > d$, if $M \in \mathcal{U}(\ell)$ satisfies $r(M) \geq f_{3.1}(r, d, \ell)$ and $\varepsilon(M) > g(r(M))$, then M has a weakly round restriction N such that $r(N) \geq r$ and $\varepsilon(N) > g(r(N))$.*

Our next lemma, proved in [8, Lemma 8.1], allows us to exploit weak roundness by contracting an interesting low-rank restriction onto a projective geometry.

Lemma 3.2. *There is an integer-valued function $f_{3.2}(n, q, t, \ell)$ so that, for any prime power q and integers $n \geq 1, \ell \geq 2$ and $t \geq 0$, if $M \in \mathcal{U}(\ell)$ is weakly round and has a $\text{PG}(f_{3.2}(n, q, t, \ell) - 1, q)$ -minor and a restriction T of rank at most t , then there is a minor N of M of rank at least n , such that T is a restriction of N , and N has a $\text{PG}(r(N) - 1, q)$ -restriction.*

4. STACKS

We now define an obstruction to $\text{GF}(q)$ -representability. If q is a prime power, and h and t are nonnegative integers, then a matroid S is a (q, h, t) -stack if there are pairwise disjoint subsets F_1, F_2, \dots, F_h of $E(S)$ such that the union of the F_i is spanning in F , and for each $i \in \{1, \dots, h\}$, the matroid $(S/(F_1 \cup \dots \cup F_{i-1}))|F_i$ has rank at most t and is not $\text{GF}(q)$ -representable. We write $F_i(S)$ for F_i . Note that such a stack has rank at most ht . When the value of t is unimportant, we refer simply to a (q, h) -stack.

The next three results suggest that stacks are ‘incompatible’ with large projective geometries. First we argue that a matroid obtained from a projective geometry by applying a small ‘projection’ does not contain a large stack:

Lemma 4.1. *Let q be a prime power and h be a nonnegative integer. If M is a matroid and $X \subseteq E(M)$ satisfies $r_M(X) \leq h$ and $\text{si}(M \setminus X) \cong \text{PG}(r(M) - 1, q)$, then M/X has no $(q, h+1)$ -stack restriction.*

Proof. The result is clear if $h = 0$; suppose that $h > 0$ and that the result holds for smaller h . Moreover suppose for a contradiction that M/X has a $(q, h+1, t)$ -stack restriction S . Let $F = F_1(S)$. Since $(M/X)|F$ is not $\text{GF}(q)$ -representable but $M|F$ is, it follows that $\square_M(F, X) > 0$. Therefore $r_{M/F}(X) < r_M(X) \leq h$ and $\text{si}(M/F \setminus X) \cong \text{PG}(r(M/F) - 1, q)$, so by the inductive hypothesis $M/(X \cup F)$ has no (q, h) -stack restriction. Since $M/(X \cup F)|(E(S) - F)$ is clearly such a stack, this is a contradiction. \square

Now we show that a large stack on top of a projective geometry allows us to find a large flat disjoint from the geometry:

Lemma 4.2. *Let q be a prime power and h be a nonnegative integer. If M is a matroid with a $\text{PG}(r(M) - 1, q)$ -restriction R and a $(q, \binom{h+1}{2})$ -stack restriction S , then $E(S) - E(R)$ contains a rank- h flat of M .*

Proof. If $h = 0$, then there is nothing to show; suppose that $h > 0$ and that the result holds for smaller h . Let S be a $(q, \binom{h+1}{2})$ -stack restriction of M and let $F_i = F_i(S)$ for each $i \in \{1, \dots, \binom{h+1}{2}\}$. Let $S_1 = S|F_1 \cup \dots \cup F_{\binom{h}{2}}$. Clearly S_1 is a $(q, \binom{h}{2})$ -stack, so inductively $E(S_1) - E(R)$ contains a rank- $(h-1)$ flat H of M .

Note that $(M/H)|E(R)$ has no loops. If M/H has a nonloop e that is not parallel to an element of R , then $\text{cl}_M(H \cup \{e\})$ is a rank- h flat of M contained in $E(S) - E(R)$, and we are done. Therefore we may assume that $\text{si}(M/H) \cong \text{si}((M/H)|E(R))$, and so by Lemma 4.1 applied to the matroid $M|(E(R) \cup H)$, we know that M/H has no (q, h) -stack restriction. However the sets $(E(S_1) - H) \cup F_{\binom{h}{2}+1}, F_{\binom{h}{2}+2}, \dots, F_{\binom{h+1}{2}}$ clearly give rise to such a stack. This is a contradiction. \square

Finally we show that a large stack restriction, together with a very large projective geometry minor, gives a projective geometry minor over a larger field:

Lemma 4.3. *There are integer-valued functions $f_{4.3}(n, q, t, \ell)$ and $h_{4.3}(n, q, \ell)$ so that, for each prime power q and any integers $\ell \geq 2$, $n \geq 2$ and $t \geq 0$, if $M \in \mathcal{U}(\ell)$ is weakly round with a $\text{PG}(f_{4.3}(n, q, t, \ell) - 1, q)$ -minor and a $(q, h_{4.3}(n, q, \ell), t)$ -stack restriction, then M has a $\text{PG}(n-1, q')$ -minor for some $q' > q$.*

Proof. Let q be a prime power and $\ell \geq 2$, $n \geq 2$ and $t \geq 0$ be integers. Let $\alpha = \alpha_{2.1}(n, q, \ell)$, and let $h' > 0$ and $r \geq 0$ be integers so that $\frac{q^{r'+h'}-1}{q-1} - q \frac{q^{2h'}-1}{q^2-1} > \alpha q^{r'}$ for all $r' \geq r$. Set $h_{4.3}(n, q, \ell) = h = \binom{h'+1}{2}$, and $f_{4.3}(n, q, t, \ell) = f_{3.2}(r + h', q, th, \ell)$.

Let $M \in \mathcal{U}(\ell)$ be weakly round with a $\text{PG}(f_{4.3}(n, q, t, \ell) - 1, q)$ -minor and an (q, h, t) -stack restriction S . We have $r(S) \leq th$; by Lemma 3.2 there is a minor N of M , of rank at least $r + h'$, with a $\text{PG}(r(N) - 1, q)$ -restriction R , and S as a restriction. By Lemma 4.2, $E(S) - E(R)$ contains a rank- h' flat F of M . Now $r(M/F) \geq r$; the lemma follows from Lemma 2.4, Theorem 2.1, and the definition of h' . \square

5. LIFTING

The following is a restatement of Theorem 1.1:

Theorem 5.1. *There is an integer-valued function $f_{5.1}(n, q, \beta)$ so that, if $\beta > 0$ is a real number, q is a prime power, and M is a $\text{GF}(q)$ -representable matroid satisfying $\varepsilon(M) \geq \beta q^{r(M)}$ and $r(M) \geq f_{5.1}(n, q, \beta)$, then M has an $\text{AG}(n - 1, q)$ -restriction.*

This next lemma uses the above to show that a bounded lift of a huge affine geometry itself contains a large affine geometry. The proof does not use the full strength of 5.1; the lemma would also follow from the much weaker ‘colouring’ Hales-Jewett Theorem [10].

Lemma 5.2. *There is an integer-valued function $f_{5.2}(n, q, \ell, t)$ so that, if $\ell \geq 2$ and $n \geq 2$ and $t \geq 0$ are integers, q is a prime power, $M \in \mathcal{U}(\ell)$ and $C \subseteq E(M)$ satisfy $r_M(C) \leq t$, and M/C has an $\text{AG}(f_{5.2}(n, q, \ell, t) - 1, q)$ -restriction, then M has an $\text{AG}(n - 1, q)$ -restriction.*

Proof. Let $\ell \geq 2$, $n \geq 2$ and $t \geq 0$ be integers, and q be a prime power. Let d be an integer large enough so that $(\ell + 1)^{-t} > \frac{q^{2-d}}{q-1}$, and let $m = f_{5.1}(n, q, (q^2(\ell + 1)^t)^{-1}) + d$. Set $f_{5.2}(n, q, \ell, t) = m$.

Let $M \in \mathcal{U}(\ell)$, and let $C \subseteq E(M)$ be a set so that $r_M(C) \leq t$ and M/C has an $\text{AG}(m - 1, q)$ -restriction R . We may assume that C is independent and that $E(M) = E(R) \cup C$, so M is simple and $r(M) = m + |C|$. Let B be a basis for M containing C , and let $e \in B - C$. Let $X = B - (C \cup \{e\})$. Now $\text{cl}_{M/C}(X)$ is a hyperplane of R , so $|\text{cl}_{M/C}(X)| = q^{m-2}$ and there are at least $q^{m-1} - q^{m-2} \geq q^{m-2}$ elements of M not spanned by $X \cup C$. Each such element lies in a point of M/X and is not spanned by C in M/X . Moreover, $r(M/X) = t + 1$, so M/X has at most $(\ell + 1)^t$ points; there is thus a point P of M/X , not spanned by C , with $|P| \geq (\ell + 1)^{-t} q^{m-2}$.

Now $P \subseteq E(R)$, so the matroid $(M/C)|P$ is $\text{GF}(q)$ -representable and has rank at most m , and $\varepsilon((M/C)|P) \geq (\ell + 1)^{-t} q^{m-2} > \frac{q^{m-d}-1}{q-1}$, so $r((M/C)|P) \geq m - d$. Furthermore, $\varepsilon((M/C)|P) \geq (q^2(\ell + 1)^t)^{-1} q^m \geq (q^2(\ell + 1)^t)^{-1} q^{r((M/C)|P)}$, so by Theorem 5.1 and the definition of m , the matroid $(M/C)|P$ has an $\text{AG}(n - 1, q)$ -restriction. However, P is skew to C in M by construction, so $(M/C)|P = M|P$, and therefore M also has an $\text{AG}(n - 1, q)$ -restriction, as required. \square

6. THE MAIN RESULT

Since, for any base- q exponentially dense minor-closed class \mathcal{M} , there is some $\ell \geq 2$ such that $\mathcal{M} \subseteq \mathcal{U}(\ell)$ and there is some s such that

$\text{PG}(s, q') \notin \mathcal{M}$ for all $q' > q$, this next theorem easily implies Theorem 1.4.

Theorem 6.1. *There is an integer-valued function $f_{6.1}(n, q, \ell, \beta)$ so that if $\beta > 0$ is a real number, $n \geq 2$ and $\ell \geq 2$ are integers, q is a prime power, and $M \in \mathcal{U}(\ell)$ satisfies $r(M) \geq f_{6.1}(n, q, \ell, \beta)$ and $\varepsilon(M) \geq \beta q^{r(M)}$, then M has either an $\text{AG}(n-1, q)$ -restriction or a $\text{PG}(n-1, q')$ -minor for some $q' > q$.*

Proof. Let $\beta > 0$ be a real number, q be a prime power, and $\ell \geq 2$ and $n \geq 2$ be integers. Let $\alpha = \alpha_{2.1}(n, q, \ell)$, $h = h_{4.3}(n, q, \ell)$. Set $0 = t_0, t_1, \dots, t_h$ to be a nondecreasing sequence of integers such that

$$t_{k+1} \geq f_{5.1}(f_{5.2}(n, q, \ell, kt_k), q, \beta((\ell+1)^{kt_k} q \alpha)^{-1})$$

for each $k \in \{0, \dots, h-1\}$. Let $m = \max(n, f_{4.3}(n, q, \ell, t_h))$, and let r_1 be an integer large enough so that $r_1 \geq (h+1)t_h$, $q^{(h+1)t_h - r_1 - 1} \leq \alpha$, and $\beta q^r \geq \alpha_{2.1}(m, q - \frac{1}{2}, \ell)(q - \frac{1}{2})^r$ for all $r \geq r_1$. Let d be an integer such that $\beta q^d \geq 1$, and let $r_2 = f_{3.1}(r_1, d, \ell)$.

Let $M_2 \in \mathcal{U}(\ell)$ satisfy $r(M_2) \geq r_2$ and $\varepsilon(M_2) \geq \beta q^{r(M_2)}$; we will show that M_2 has either a $\text{PG}(n-1, q')$ -minor for some $q' > q$, or an $\text{AG}(n-1, q)$ -restriction. The function $g(r) = \beta q^r$ satisfies $g(d) \geq 1$ and $g(r) \geq 2g(r-1)$ for all $r > d$, so by Lemma 3.1 M_2 has a weakly round restriction M_1 such that $r(M_1) \geq r_1$ and $\varepsilon(M_1) \geq \beta q^{r(M_1)}$.

Let $k \geq 0$ be maximal such that $k \leq h$ and M_1 has a (q, k, t_k) -stack restriction S . We split into cases depending on whether $k = h$:

Case 1: $k = h$.

Note that $\varepsilon(M_1) \geq \beta q^{r(M_1)} \geq \alpha_{2.1}(m, q - \frac{1}{2}, \ell)(q - \frac{1}{2})^{r(M_1)}$, so M_1 has a $\text{PG}(m-1, q')$ -minor for some $q' > q - \frac{1}{2}$. If $q' > q$, then we have the first outcome, since $m > n$. Therefore we may assume that M_1 has a $\text{PG}(m-1, q)$ -minor. M_1 also has a (q, h, t_h) -stack restriction, and the first outcome now follows from Lemma 4.3 and the definitions of m and h .

Case 2: $k < h$.

Let $M_0 = \text{si}(M_1/E(S))$; note that $r(M_0) \geq r(M_1) - kt_k$, and therefore that $|M_0| \geq (\ell+1)^{-kt_k} |M_1| \geq (\ell+1)^{-kt_k} \beta q^{r(M_0)}$. Let F_0 be a rank- $(t_{k+1}-1)$ flat of M_0 , and consider the matroid M_0/F_0 . If $\varepsilon(M_0/F_0) \geq \alpha q^{r(M_0/F_0)}$, then we have the first outcome by Theorem 2.1, so we may assume that $\varepsilon(M_0/F_0) \leq \alpha q^{r(M_0/F_0)} = \alpha q^{r(M_0) - t_{k+1} + 1}$. Let \mathcal{F} be the collection of rank- t_{k+1} flats of M_0 containing F_0 . By a majority

argument, there is some $F \in \mathcal{F}$ satisfying

$$\begin{aligned} |F - F_0| &\geq |\mathcal{F}|^{-1} |M_0 \setminus F_0| \\ &\geq \varepsilon(M_0/F_0)^{-1} ((\ell + 1)^{-kt_k} \beta q^{r(M_0)} - |F_0|) \\ &\geq \alpha^{-1} q^{-r(M_1)+t_{k+1}-1} ((\ell + 1)^{-kt_k} \beta q^{r(M_0)} - |F_0|). \end{aligned}$$

Since $\alpha^{-1} q^{-r(M_1)+r(S)+t_{k+1}-1} \leq \alpha^{-1} q^{-r_1+(h+1)t_h-1} \leq 1$, this gives $|F| \geq \alpha^{-1} q^{-r(M_0)+t_{k+1}-1} (\ell + 1)^{-kt_k} \beta q^{r(M_0)} = \beta ((\ell + 1)^{kt_k} q \alpha)^{-1} q^{r(M_0|F)}$. By maximality of k , we know that $M_0|F$ is $\text{GF}(q)$ -representable, and $r(M_0|F) = t_{k+1} \geq f_{5.1}(f_{5.2}(n, q, \ell, kt_k), q, \beta((\ell + 1)^{kt_k} q \alpha)^{-1})$, so $M_0|F$ has an $\text{AG}(f_{5.2}(n, q, \ell, kt_k) - 1, q)$ -restriction by Theorem 5.1. Now $M_0 = \text{si}(M_1/E(S))$ and $r(S) \leq kt_k$, so by Lemma 5.2, M_1 has an $\text{AG}(n - 1, q)$ -restriction, and so does M_2 . \square

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